High-order Sliding-Mode Based Sub-optimal Linear Quadratic Regulator with Application to Roll Autopilot Design

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Abstract. A sub-optimal robust controller is designed for linear systems affected by external disturbances or bounded uncertainties. The sub-optimal controller is composed by two terms: first, a linear quadratic regulator, that provides optimal stabilization, is designed for the linear system in the absence of perturbations; second, a nonlinear compensation term, designed using the high-order sliding-modes techniques, is used to compensate the perturbations that affect the linear system. The proposed sub-optimal controller is applied to the design of a roll autopilot for a missile.

Keywords: Roll control, high-order sliding-modes, sub-optimal control

1 Introduction

1.1 State of the art

The design of controllers for perturbed systems is one of the most challenging problems in control theory. The control of systems under uncertain conditions has been successfully addressed using adaptive and robust techniques. However, for the aerospace applications, the appearance of other important phenomena during the flight significantly complicate the application of the adaptive control techniques. The fast changes in the plant's parameters caused by a fault must be identified and the control law reconfigured online. For example, both, the control derivatives and the stability derivatives, undergo significant changes due to a control surface fault, and control surface failure causes a trim disturbance that needs to be rejected by the flight control system [1].

Sliding mode control is known as an effective technique to deal with perturbed or uncertain systems, the application of sliding mode control techniques is restricted by the appearance of the chattering effect [2] (an undesirable high-frequency oscillation appearing on the system variables).

The High-Order Sliding-Modes techniques (see, for example, [3, 4]) mitigates the application problems related to standard sliding-modes by reducing the switching frequency needed to maintain the sliding motion. This High-Order Sliding-Mode control techniques have been, in general, designed to stabilize systems with a relative degree greater than one preserving the robustness and accuracy of the standard sliding-modes.

The High-order sliding-modes techniques have already been applied to flight control problems with satisfactory results. In [6] a sliding modes based control is designed using two control loops, the proposed control ensures asymptotic tracking of the command deflections. The smooth second-order sliding-modes is applied for missile guidance in [7]. The dynamic sliding-manifolds technique is applied in combination with a transformation for stabilization of nonminimum phase aircrafts in the work by [8]. A fault detection algorithm and a fault tolerant control for a large aircraft with specific application in a B747 simulation model is presented in [9]. Recently [10] developed a 2nd order sliding-modes based black-box control for signal tracking, the proposed controller is tested in simulations with a 6-DOF UAV model.

An alternative to obtain the characteristic robustness of the sliding-modes, without applying directly the discontinuous control signals on the system, is the use of these techniques for the design of estimation algorithms. Disturbance identification algorithms are usually applied for fault tolerant and robust control design and the High-Order Sliding-Modes have been successfully applied for the design of observers and algorithms for estimation of disturbances (see, for example, [11, 12]). The main advantages of the observers basing on high-order sliding-modes is their robustness against external perturbations [13–16], and that they bring the possibility of exploit the equivalent output injection for the designing of the identification algorithms for the disturbances.

The combination of sliding-mode control techniques with conventional control has allowed the development of robust control algorithms that are capable to solve the stabilization problem under uncertain conditions. Some of these works are briefly described below. In [17] a Linear Quadratic Regulator is applied to stabilize a nonlinear affine system using a compensation term generated by the use of integral sliding modes. In [18] the high-order sliding mode based hierarchical observer is applied to identify disturbances acting on a perturbed system, the identified signal is used to add robustness to the smooth control signal generated by standard feedback control. A backstepping design that combines the high-order sliding modes differentiator and the feedback linearization is proposed in [19].

In [20] the general model of an autopilot for tactical missiles is proposed. In this article, the dynamic of the missile is spliced into two decoupled dynamics. In one side the first order rigid body effect is considered; while by the other side, the dynamic corresponding to the flexible body dynamics is considered. Using this model, in [21] a robust autopilot is proposed using a Linear Quadratic Regulator with a compensation term designed using the methodology proposed in [22].

1.2 Main contribution

In this paper, a sub-optimal robust controller is designed for linear systems affected by external disturbances. The proposed controller is composed by two terms:

- A linear quadratic regulator capable to perform optimal stabilization of the linear system in the absence of perturbations.
- A nonlinear compensation term, designed using the high-order sliding-mode observer, that is used to compensate the perturbations that affect the linear system.

The controller is used to design a roll autopilot for a missile. The performance of the proposed observer is illustrated by simulations in the model presented in [20] and [21].

1.3 Paper structure

In Section 2, the class of systems under study is defined. The robust Linear Quadratic Regulator is designed in section 3, in particular, the linear part of the controller is presented in Subsection 3.1, and the high-order sliding-modes based compensation term is designed in Section 3.2. The application of the proposed technique to the roll autopilot design is given in Section 4. Section 5 provides conclusions to this study.

2 Problem statement

Let consider the following perturbed linear system

$$\dot{x} = Ax + Bu + Df \tag{1}$$
$$y = Cx \tag{2}$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^p$ is the system output, $u \in \mathbb{R}^m$ and $f \in \mathbb{R}^q$ are the control signal and disturbances, respectively. The matrices A, B, C, and D are all conformable matrices.

The Rosenbrock matrix of the triplet $\{A, C, D\}$ is defined as:

$$R(s) = \begin{bmatrix} sI_n - A & -D \\ C & 0 \end{bmatrix} \tag{3}$$

The invariant zeros of the triplet $\{A, C, D\}$ are given by the points s_0 for which the Rosenbrock matrix $R(s_0)$ loses rank.

It is considered that the system (1)-(2) satisfies the following assumptions:

Assumption 1 *The triplet $\{A, C, D\}$ does not have invariant zeros.*

Assumption 2 *The perturbation signal f satisfies*

$$\|f\|_\infty \leq f^+$$

for a known scalar $f^+ > 0$, here $\|\cdot\|_\infty$ denotes the infinite norm.

Assumption 3 *The control distribution matrix B and the disturbance term Df satisfy:*

$$Df \in spanB$$

Differential equations are understood in the Filippov sense [23] in order to provide for the possibility to use discontinuous signals in controls. Filippov solutions coincide with the usual solutions, when the right-hand sides are Lipschitzian. It is assumed also that all considered inputs allow the existence of solutions and their extension to the whole semi-axis $t \geq 0$.

The aim of this paper is designing a Linear Quadratic Regulator algorithm that can stabilize the state of the system (1)-(2) even in the presence of the disturbances vector f.

3 Robust Linear Quadratic Regulator Design

The proposed controller is composed by two signals. The first one, is used to provide for optimal stabilization the nominal system, while the second one is designed to guarantee robustness against the perturbation f. The control takes the following form

$$u = u_1 + u_2 \tag{4}$$

where u_1 and u_2 will be designed below.

3.1 Linear Quadratic Regulator

The Linear Quadratic Regulator is designed to minimize a quadratic performance index of the form

$$J = \int_0^\infty \left(x^T Q x + u_1^T R u_1\right) dt$$

where $Q \geq 0$ and $R > 0$ are weights to be chosen. The resulting is an optimal control law given by:

$$u_1 = -Kx \tag{5}$$

where the gain K is computed as $K = R^{-1}B^T P$, where P is computed as the solution of the matrix algebraic Riccatti equation

$$PA + A^T P - PBR^{-1}B^T P + Q = 0$$

By an appropriate selection of matrixes Q and R, one can obtain the desired performance. This optimal controller is usually applied to solve a wide variety of problems, as for example, this controller is an usual tool for the design of autopilots.

3.2 High Order Sliding Modes based compensator

Under Assumption 1 the system (1),(2) is strongly observable (to see a deeper study about strong detectability, the reader can refer, for example, to the tutorial book [24]). This assumption allows us to reconstruct exactly and in a finite-time the state, even in the presence of the disturbance f [14].

With this aim, an observer which is based on the high-order sliding modes is proposed as:

$$\begin{aligned} \dot{z} &= Az + Bu + L(y - y_z) \\ e_y &= y - y_z \\ \hat{x} &= z + U^{-1}v(e_y) \end{aligned} \tag{6}$$

where the matrix U takes the form

$$U = \begin{bmatrix} C \\ C(A-LC) \\ \vdots \\ C(A-LC)^{n-1} \end{bmatrix}$$

the compensation term $v(e_y)$ is composed by the variables

$$v(e_y) = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

where the components of the vector v_i $i = 1, ..., n$, and the additional variable v_{n+1} are taken from the high order sliding mode differentiator [25] given by:

$$\begin{aligned} \dot{v}_1 &= w_1 \\ w_1 &= -\alpha_{n+1} M^{1/(n+1)} |v_1 - e_y|^{n/(n+1)} \operatorname{sign}(v_1 - e_y) + v_2 \\ \dot{v}_2 &= w_2 \\ w_2 &= -\alpha_n M^{1/n} |v_2 - w_1|^{(n-1)/n} \operatorname{sign}(v_2 - w_1) + v_3 \\ &\vdots \\ \dot{v}_n &= w_n \\ w_n &= -\alpha_2 M^{1/2} |v_n - w_{n-1}|^{1/2} \operatorname{sign}(v_n - w_{n-1}) + v_{n+1} \\ \dot{v}_{n+1} &= -\alpha_1 M \operatorname{sign}(v_{n+1} - w_n) \end{aligned} \tag{7}$$

where the parameter M is chosen sufficiently large, in particular $M > |d|f^+$, where $d = C(A-LC)^{n-1}D$. The constants α_i are chosen recursively sufficiently large as in [25]. In particular, one of the possible choices is $\alpha_1 = 1.1$, $\alpha_2 = 1.5$, $\alpha_3 = 2$, $\alpha_4 = 3$, $\alpha_5 = 5$, $\alpha_6 = 8$, which is sufficient for $n \leq 6$. Note that (7) has a recursive form, useful for the parameter adjustment. In any computer realization one has to calculate the internal auxiliary variables v_j and w_j, $j = 1, ..., n$, using only the simultaneously-sampled current values of e_y and v_j.

The auxiliary output estimation error e_y and its first n derivatives take the following form

$$\begin{aligned} e_y &= y - Cz = C(x - z) \\ \dot{e}_y &= C(A-LC)(x - z) \\ &\vdots \\ e_y^{(n)} &= C(A-LC)^n(x - z) + C(A-LC)^{n-1}Df \end{aligned}$$

On the other hand, the high order sliding mode differentiator (7) brings an estimation of the derivatives up to order $n-1$. Hence, after the convergence of the differentiator, the derivative of order n satisfies:

$$-\alpha_2 M^{1/2}|v_n - w_{n-1}|^{1/2}\operatorname{sign}(v_n - w_{n-1}) + v_{n+1} = C(A-LC)^n(x-z) + C(A-LC)^{n-1}Df$$

Thus, the following equality holds after a finite time transient

$$v_{n+1} = C(A-LC)^n(x-z) + C(A-LC)^{n-1}Df. \tag{8}$$

The equation (8) is called the equivalent output injection. Given the properties of the differentiator (7), v_{n+1} is a continuous term.

The perturbation f can be identified through the equivalent output injection as:

$$\hat{f} = \left(C(A-LC)^{n-1}D\right)^{-1}\left(v_{n+1} - C(A-LC)^n U^{-1} v(e_y)\right)$$

Notice that $C(A-LC)^{n-1}D \neq 0$ otherwise no perturbations affects the system.

The control term that provides robustness against the disturbance f is proposed as

$$u_2 = H\hat{f} \tag{9}$$

where the matrix H is computed as $H = B^+D$, the matrix B^+ is the Moore-Penrose left pseudoinverse of B, i.e., $B^+ = (B^TB)^{-1}B^T$.

Theorem 1. *Be the system (1)-(2). Under Assumptions 1 - 3 the controller (4) provides suboptimal exact regulation under the presence of external perturbations.*

4 Application to the Robust Roll Autopilot Design

A wide variety of missiles possesses a cruciform configuration which brings to them a high accuracy and quick manoeuvering in any direction. However, the inherent instability of the roll yields in undesirable rolling motions that degrades the performance. To overcome this problem, the roll autopilots are proposed (see [21, 20]). The main objective of the above mentioned controllers is to maintain the attitude of the missile under system variations and external disturbances.

The block diagram of the missile roll dynamic is shown in Figure 1. The airframe flexibility is considered in the flexible body dynamics block. External disturbances d_{ext} are used to describe external perturbations.

The control is designed disregarding the flexible body effects and the external disturbances. In this sense, for analysis proposes, the flexible body dynamics, external perturbations and any other coupling effect derived from the pitch and yaw motions are concentrated in a single term, f. The state equations of the system can be written as:

$$\dot{x} = Ax + Bu + Df$$
$$y = Cx$$

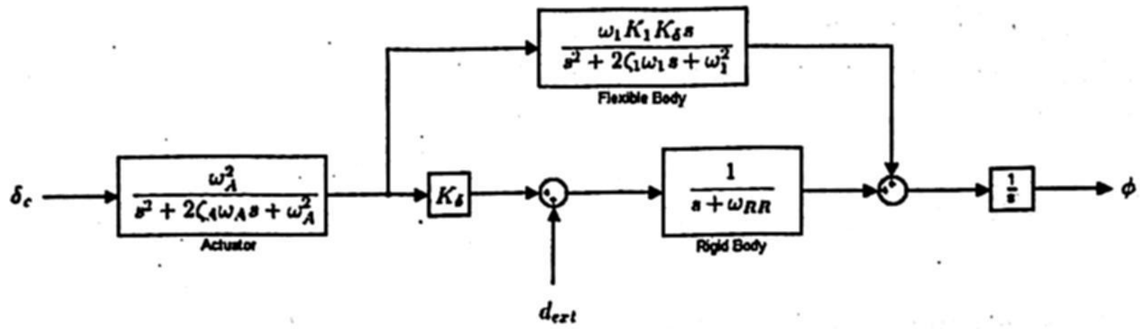

Fig. 1. System block diagram.

where the matrices A, B and C take the following form

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 & -a_0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ b \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} 1\,0\,0\,0 \end{bmatrix}$$

where $a_0 = 2\zeta_A\omega_A + \omega_{RR}$, $a_1 = 2\zeta_A\omega_A\omega_{RR} + \omega_A^2$ $a_2 = \omega_A^2\omega_{RR}$, $a_3 = 0$, $b = \omega_A^2 K_\delta$.

The parameters are given in the following table:

Symbol	Variable	Value
ω_{RR}	Roll rate bandwidth	2 rad/s
K_δ	Fin Effectiveness	9000 $1/s^2$
ω_A	Actuator bandwidth	100 rad/s
ζ_A	Actuator damping	0.65
ω_1	Torsional mode frequency	250 rad/s
ζ_1	Torsional mode damping	0.01
K_1	Torsional mode gain	-0.0000129
ϕ_{max}	Maximum desired roll angle	10
$\dot{\phi}_{max}$	Maximum desired roll rate	300 deg/s
$\delta_{c(max)}$	Maximum desired fin deflection	30 deg

Table 1. System parameters.

For simulation purposes, the perturbation is given by:

$$d_{ext} = 54600 + 50000\sin(\sin(t)\sin(0.1t) + 0.2)$$

The weighting matrices Q and R are the same as in [20]:

$$Q = \begin{bmatrix} \frac{1}{\phi_{max}^2} & 0 & 0\,0 \\ 0 & \frac{1}{\dot{\phi}_{max}^2} & 0\,0 \\ 0 & 0 & 0\,0 \\ 0 & 0 & 0\,0 \end{bmatrix} \quad R = \begin{bmatrix} \frac{1}{\delta_{c(max)}^2} \end{bmatrix}$$

where ϕ_{max}, $\dot{\phi}_{max}$, $\delta_{c(max)}$ are the maximal permissible values of the respective variables.

Using the solution of the algebraic Matrix Riccatti equation, the gain K for the controller (5) is given by

$$K = \begin{bmatrix} 3 & 0.1286 & 0.001 & 0 \end{bmatrix}$$

The eigenvalues of matrix A are 0, -2, $-65+75.9934i$, $-65-75.9934i$, notice that the system is marginally stable. The Luenberger gain of the observer is designed to obtain a stable estimation error. The gain L is chosen to place the roots of the estimation error dynamics matrix $(A-LC)$ in -40, -41, -42, -43, as

$$L = \begin{bmatrix} 34 \\ -4417 \\ 499890 \\ -18385220 \end{bmatrix}$$

Using the compensated system matrix $(A-LC)$, the matrix U is given by:

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -34 & 1 & 0 & 0 \\ 5573 & -34 & 1 & 0 \\ -839550 & 5573 & -34 & 1 \end{bmatrix}$$

The high-order sliding mode differentiator takes the form:

$$\begin{aligned}
\dot{v}_1 &= w_1 = -\alpha_5 M^{1/5}|v_1 - e_y|^{4/5}\,\mathrm{sign}(v_1 - e_y) + v_2 \\
\dot{v}_2 &= w_2 = -\alpha_4 M^{1/4}|v_2 - w_1|^{3/4}\,\mathrm{sign}(v_2 - w_1) + v_3 \\
\dot{v}_3 &= w_3 = -\alpha_3 M^{1/3}|v_3 - w_2|^{2/3}\,\mathrm{sign}(v_3 - w_2) + v_4 \\
\dot{v}_4 &= w_4 = -\alpha_2 M^{1/2}|v_4 - w_3|^{1/2}\,\mathrm{sign}(v_4 - w_3) + v_5 \\
\dot{v}_5 &= -\alpha_1 M\,\mathrm{sign}(v_5 - w_4)
\end{aligned}$$

where the gains are chosen as $\alpha_1 = 1.1$, $\alpha_2 = 1.5$, $\alpha_3 = 2$, $\alpha_4 = 3$, $\alpha_5 = 5$ and $M = 2000$.

The control signal (4) is given by $u = -K\hat{x} + H\hat{f}$, where

$$H = \begin{bmatrix} 0 & 0 & 0 & 0.1111 \times 10^{-7} \end{bmatrix}$$

The convergence of the roll angle ϕ to zero after a finite time transient is shown in the Figure 2. Deflection of the ailerons δ and its ratio are presented in the Figure 3. The perturbation identification $\hat{w}$ is shown in Figure 4.

The results obtained with the proposed methodology are compared with a standard Linear Quadratic Regulator. With this aim, the control signal takes the form:

$$u = -Kx$$

The roll angle and a zoom on the graphic using the standard Linear Quadratic Regulator are shown in the Figure 5. In comparison with the standard Linear Quadratic Regulator, the robust Linear Quadratic Regulator is exact with

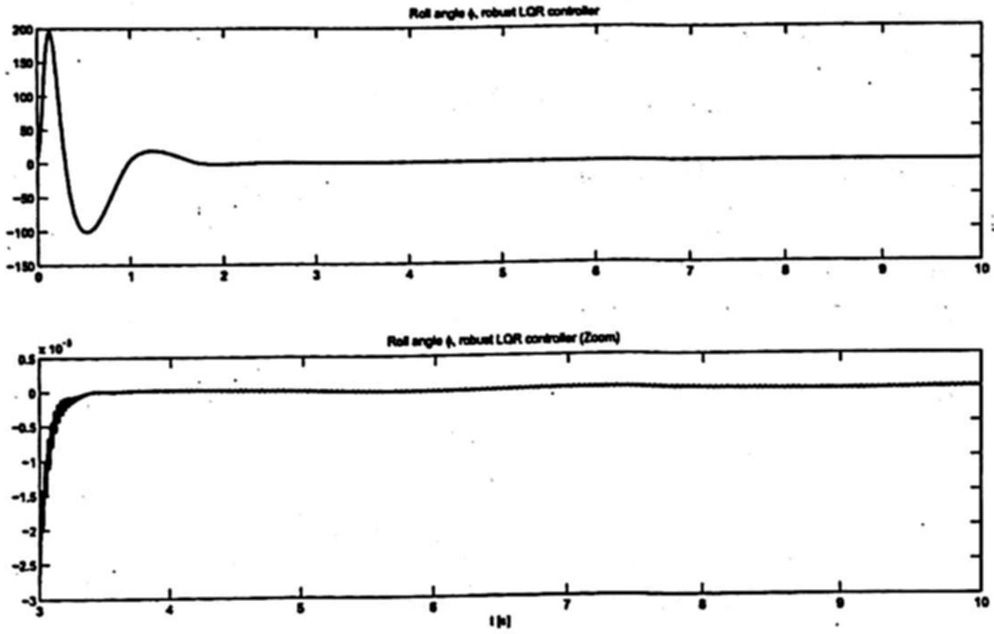

Fig. 2. Roll angle ϕ (above) and a zoom on the image (below) for the robust LQR controller.

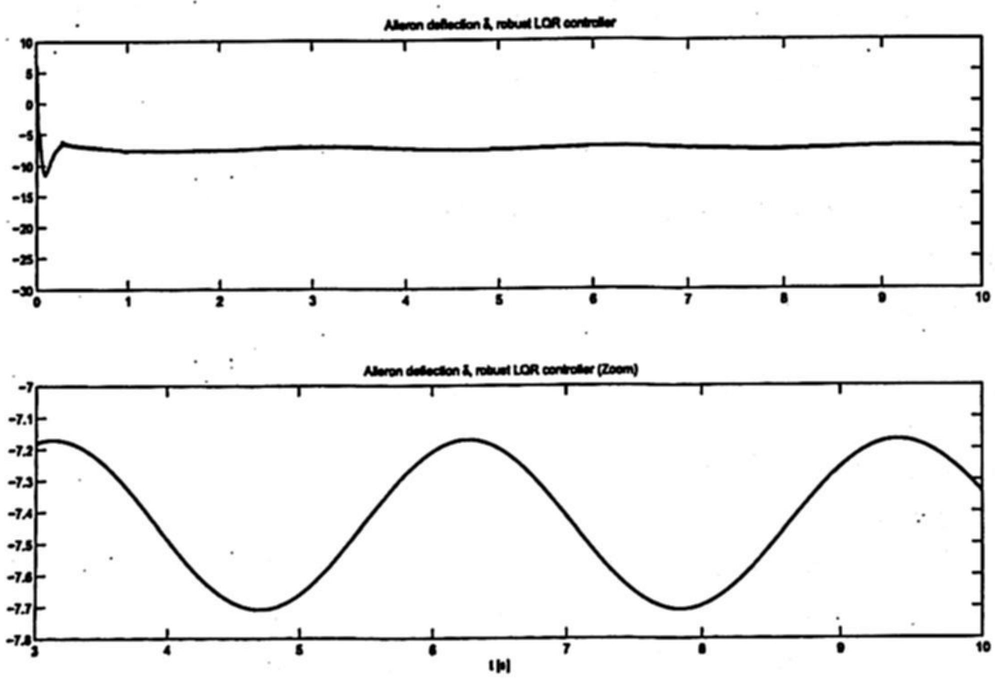

Fig. 3. Aileron Deflections δ for the robust LQR controller.

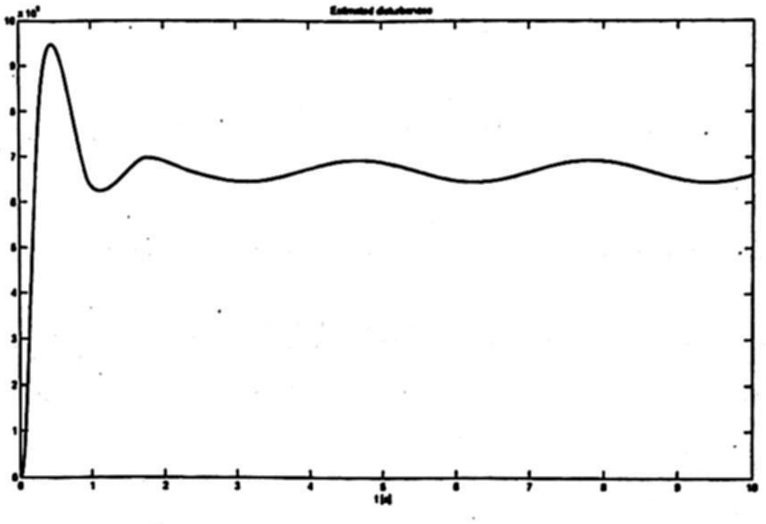

Fig. 4. Estimated disturbances.

respect to the coordinate ϕ. The aileron deflection δ for the standard Linear Quadratic Regulator is shown in the Figure 6. Notice that after 2.5 seconds, the deflections obtained for both controllers are very similar, then the most important contribution of the nonlinear compensation term takes place during the transient.

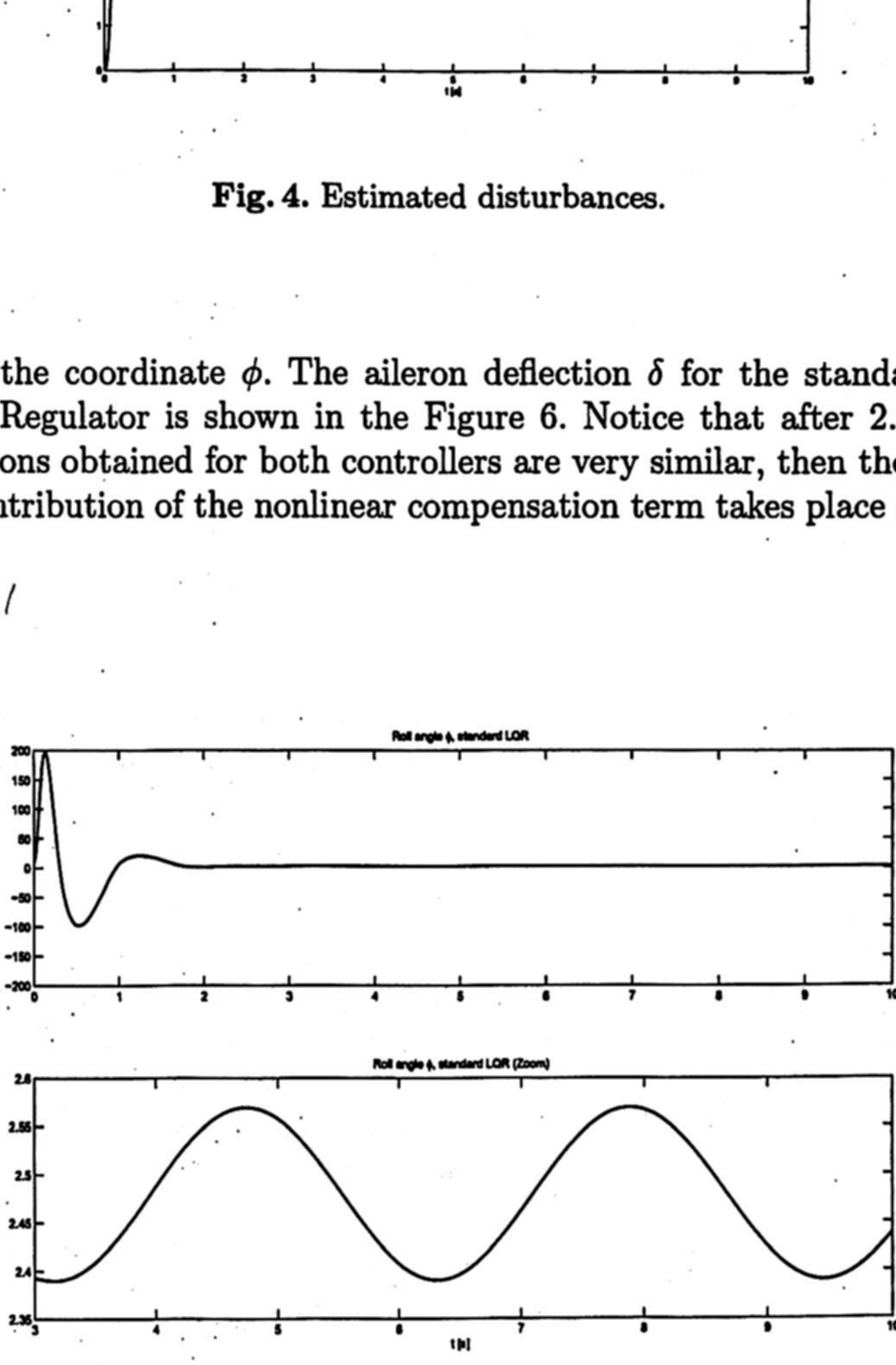

Fig. 5. Roll angle ϕ (above) and a zoom on the image (below) for the standard LQR controller.

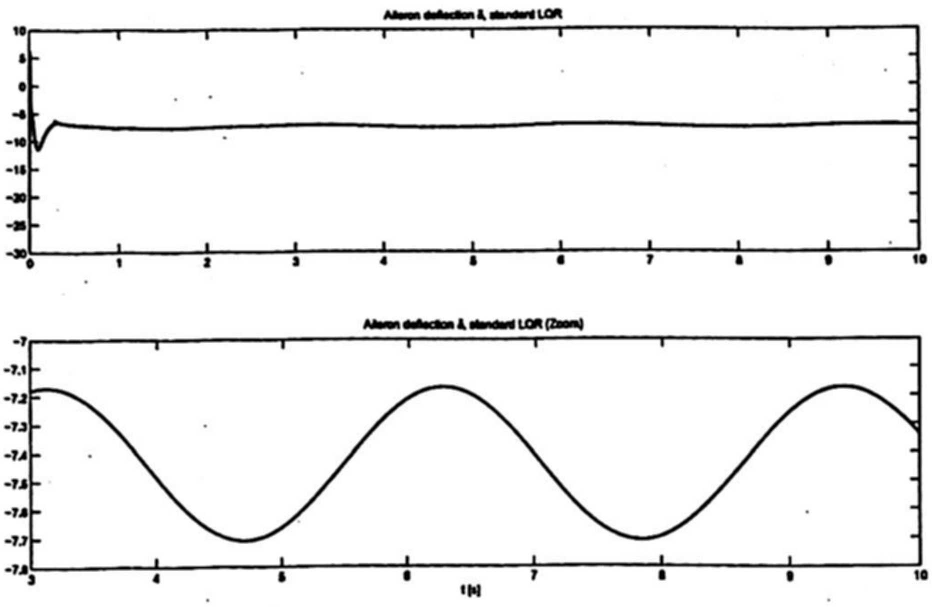

Fig. 6. Aileron Deflections δ (above) and a zoom on the image (below) for the standard LQR controller.

5 Conclusions

In this paper a sub-optimal robust controller is proposed for linear systems. The controller is composed of two terms. The linear quadratic regulator term allowed the optimal stabilization of the linear system in the absence of perturbations, while the high-order sliding-mode based compensation term compensates the effect of perturbations disrupting the system. The proposed method is used to design a roll autopilot for a missile model. The robustness of the proposed controller is illustrated by simulations.

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